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Short Communication

Maximal Lyapunov exponent and almost-sure stability for Stochastic Mathieu–Duffing Systems ☆

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Abstract

The principal resonance of a Mathieu–Duffing oscillator to random excitation is investigated. The method of multiple scales is used to determine the equations of modulation of amplitude and phase. The effects of damping, detuning, cubic term, and magnitudes of random excitation are analyzed. The explicit asymptotic formulas for the maximal Lyapunov exponent is obtained. The almost-sure stability or instability of the stochastic Mathieu–Duffing system depends on the sign of the maximal Lyapunov exponent. In the last part of the work, the numerical results are obtained.

1. Introduction

Consider the Mathieu–Duffing system which described by the following ordinary differential equation:

$$\ddot{x} + \varepsilon h\dot{x} + \omega_0^2 x + \varepsilon\beta x \cos(\Omega t) + \alpha x^3 = 0. \tag{1}$$

The equation is extensively used for parametric nonlinear vibrations in engineering. It is important to investigate dynamical behaviors of this system. For weak excitations, Yamaguchi [1]

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in 1985 investigated the structure of stochastic layer for such an oscillator through the Chirikov overlap. Esmailzadeh and Nakhaie-Jazar [2] in 1997 found that there exists necessary and sufficient conditions for the existence of at least one periodic solution for the Mathieu–Duffing equation. Natsiavas et al. [3,4] investigated a Mathieu–Duffing oscillator under constant external load. They reveals that the oscillator examined may exhibit strong (first-order) resonance for two ranges of the forcing frequency. The first one occurs when the excitation frequency, it has identical form with the Mathieu–Duffing oscillator under principal parametric resonance. However, unlike the Mathieu–Duffing oscillator with no constant external forcing, the system examined exhibit a second strong resonance, which occurs in forcing frequency ranges near the linear natural frequency. Luo [5,6] in 2003 investigated Mathieu–Duffing oscillator with a twin-well potential, in his work, the approximate criteria for the onset and destruction of a specified, primary resonant band of the Mathieu–Duffing oscillator was developed. Ng et al. [7,8] investigated the Mathieu equation to which is added a cubic nonlinearity x^3 term using averaging method. However, all above work is due to a deterministic Mathieu–Duffing oscillator. Rong Haiwu et al. [9] in 2003 investigated the almost-sure stability or instability of the stochastic Mathieu system depends on the sign of the maximal Lyapunov exponent.

In this paper, we will investigate the almost-sure stability or instability of the Mathieu–Duffing under the stochastic parametric excitation by stochastic multiple scales method.

2. Multiple scales method

Consider the following Mathieu–Duffing oscillator under random excitation:

$$\ddot{x} + \varepsilon h\dot{x} + \omega_0^2 x + \varepsilon\beta x \cos(\Omega t + \gamma W(t) + \delta) + \varepsilon\alpha x^3 = 0, \tag{2}$$

where dots indicate differentiation with respect to the time t, ε is a small parameter, ω_0^2 and h are natural frequency and damping coefficient, respectively, $W(t)$ is a wiener process, Ω is the excitation frequency, γ is excitation strength, δ is a random variable uniformly distributed in $[0, 2\pi]$. The system under random excitation is called stochastic Mathieu–Duffing oscillator.

The method of multiple scales [10] has been widely used in the analysis of deterministic systems. Nayfeh and Serhan [11] extend this method to the analysis of nonlinear systems under random external excitation. Rong Haiwu et al. [12] extend this method to the nonlinear systems under random internal excitation. In this paper, the multiple scales method is used to investigate the response and stability of system (2). Then, a uniformly approximate solution of Eq. (2) is sought in the form

$$x(t, \varepsilon) = x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) + \varepsilon^2 x_2(T_0, T_1) + \cdots, \tag{3}$$

where $T_0 = t$, $T_1 = \varepsilon t$ are fast and slow time scale, respectively.

Throughout this paper, we only discuss the first-order uniform expansion of the solution $x_0(T_0, T_1)$ of Eq. (2). By denoting $D_n = \partial/\partial T_n$ $(n = 0, 1, 2, \ldots, m)$, the ordinary-time derivatives can be transformed into partial derivatives as

$$\frac{\mathrm{d}}{\mathrm{dt}} = D_0 + \varepsilon D_1 + \cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \cdots . \tag{4}$$

Substituting Eqs. (3) and (4) into Eq. (2) and comparing coefficients of ε with equal powers, one obtains the following equations:

$$D_0^2 x_0 + \omega_0^2 x_0 = 0, \tag{5}$$

$$D_0^2 x_1 + \omega_0^2 x_1 = -2D_0 D_1 x_0 - hD_0 x_0 - \beta x_0 \cos\varphi - \alpha x_0^3, \tag{6}$$

where $\varphi = \Omega t + \gamma W(t) + \delta$. The general solution of Eq. (5) can be written as

$$x_0 = A(T_1)\exp(\mathrm{i}\omega_0 T_0) + \bar{A}(T_1)\exp(-\mathrm{i}\omega_0 T_0). \tag{7}$$

We shall investigate the principal resonance of system (2), it is clear that resonance occurs when $\Omega - 2\omega_0 = o(\varepsilon)$. Introducing the detuning parameter σ as follows: $\Omega = 2\omega_0 + \varepsilon\sigma$. Substituting Eq. (7) into Eq. (6), one obtains

$$\begin{aligned} D_0^2 x_1 + \omega_0^2 x_1 = {} & -\frac{\beta}{2}\bar{A}(T_1)\exp(\mathrm{i}\omega_0 T_0 + \mathrm{i}\sigma t + \mathrm{i}\gamma W(T_1) + \mathrm{i}\delta) - 2\mathrm{i}\omega_0 A' \exp(\mathrm{i}\omega_0 T_0) \\ & - hA(T_1)\mathrm{i}\omega_0 \exp(\mathrm{i}\omega_0 T_0) - \alpha A^3 \exp(3\mathrm{i}\omega_0 T_0) - 3\alpha A^2 \bar{A} \exp(\mathrm{i}\omega_0 T_0) + \mathrm{cc}, \end{aligned} \tag{8}$$

where cc represents the complex conjugate of its preceding terms. Then, elimination of the secular terms yields

$$2\mathrm{i}\omega_0 A' + \frac{\beta}{2}\bar{A}(T_1)\exp(\mathrm{i}\sigma T_1 + \mathrm{i}\gamma W(T_1) + \mathrm{i}\delta) + 3\alpha A^2\bar{A} + hA(T_1)\mathrm{i}\omega_0 = 0, \tag{9}$$

where the prime stands for the derivative with respect to T_1. Expressing A in the polar form

$$A(T_1) = a(T_1)\exp(\mathrm{i}\theta(T_1)). \tag{10}$$

Substituting Eq. (10) into Eq. (9), one obtains

$$a' = -\frac{\beta}{4\omega_0} a \sin\eta - \frac{h}{2} a,$$

$$a\theta' = \frac{\beta}{4\omega_0} a \cos\eta + \frac{3\alpha}{2\omega_0} a^3, \tag{11}$$

where $\eta = \sigma T_1 + \gamma W(T_1) + \delta - 2\theta(T_1)$. Eq. (11) can be written as

$$a' = -\frac{\beta}{4\omega_0} a \sin\eta - \frac{h}{2} a,$$

$$a\eta' = \sigma a - \frac{\beta a}{2\omega_0}\cos\eta - \frac{3\alpha}{\omega_0} a^3 + a\gamma W'(T_1). \tag{12}$$

After solving a and η, the first-order uniform expansion of the solution of Eq. (2) is given by

$$x = 2a(\varepsilon t)\cos\left[\frac{\Omega}{2} t - \frac{\eta(\varepsilon t)}{2}\right] + o(\varepsilon).$$

3. Stability and Lyapunov exponents

By Eq. (12), we can conclude the nonlinear term x^3 take effects on the amplitude of the system's response by influencing the phase η. When $\sin\eta < -2h\omega_0/\beta$ (i.e., $a' > 0$), there are energy input in the system so amplitude a will be increased and η be changed. The system takes on stable vibration and $a' = 0$, $\eta' = 0$ when velocity of input energy equals velocity of output energy.

3.1. Trivial solution and its stability

Obviously $a = 0$ is the trivial solution of Eqs. (12), now we discuss its stability. We obtain the following linearized equations of (12) at $(0, 0)$

$$a' = -\frac{\beta}{4\omega_0} a \sin\eta - \frac{h}{2} a,$$

$$\eta' = \sigma - \frac{\beta}{2\omega_0}\cos\eta + \gamma W'(T_1). \tag{13}$$

Let $v = \ln a$, Eq. (13) can be written as the following Ito equations:

$$\mathrm{d}v = \left(-\frac{h}{2} - \frac{\beta}{4\omega_0}\sin\eta\right)\mathrm{d}T_1,$$

$$\mathrm{d}\eta = \left(\sigma - \frac{\beta}{2\omega_0}\cos\eta\right)\mathrm{d}T_1 + \gamma\,\mathrm{d}W(T_1), \tag{14}$$

η steady-state probability density function obtained by detail balance method is given as

$$p(\eta) = \frac{\exp[\bar{\sigma}\eta + \bar{\beta}\cos\eta]}{C}\int_{\eta}^{\eta+2\pi}\exp[-(\bar{\sigma}x + \bar{\beta}\cos x)]\,\mathrm{d}x, \tag{15}$$

where C is the normalization constant of the invariant measure and is given by

$$C = 4\pi^2\exp(-\bar{\sigma}\pi) I_{\mathrm{i}\bar{\sigma}}(\bar{\beta}) \cdot I_{-\mathrm{i}\bar{\sigma}}(\bar{\beta}) \tag{16}$$

and $I_n(x)$ is the Bessel function of the first kind and n can be any real and complex number.

$\bar{\sigma} = 2\sigma/\gamma^2$, $\bar{\beta} = \beta/\omega_0\gamma^2$. The largest Lyapunov exponent $\lambda = \lambda_{\max}$ of the corresponding solution (a_0, η_0) is given by

$$\lambda = (a_0, \eta_0)\lim_{T_1\to\infty}\frac{1}{T_1}\ln|a(T_1; a_0, \eta_0)|, \quad \text{w.p. 1.} \tag{17}$$

The almost certain stability of the trivial solution (13) can be determined by the largest Lyapunov exponent $\lambda = \lambda_{\max}$, when $\lambda < 0$ the trivial solution is almost certainly stable and when $\lambda > 0$ the trivial one is almost certainly unstable, hence $\lambda = 0$ is the bifurcation point of the stability of the trivial solution. From Eqs. (17), one has

$$\begin{aligned}\lambda &= \lim_{T_1\to\infty}\frac{1}{T_1}\ln\left|\frac{a(T_1)}{a(T_0)}\right| = \lim_{T_1\to\infty}\frac{1}{T_1}(\ln a(T_1)-\ln a(T_0)) \\ &= \lim_{T_1\to\infty}\frac{1}{T_1}(v(T_1)-v(T_0)) = \lim_{T_1\to\infty}\frac{1}{T_1}\int_0^{T_1}\mathrm{d}v \\ &= -\frac{h}{2}-\frac{\beta}{4\omega_0}\int_0^{2\pi}\sin\eta\cdot p(\eta)\,\mathrm{d}\eta. \qquad (18)\end{aligned}$$

When $h = 0.0$, $\omega_0 = 1.0$, $\gamma = 0.5$, the variation of λ with σ and β are shown in Fig. 1(a), and the corresponding level lines are shown in Fig. 1(b).

Fig. 1(a) and (b) show the three-dimensional plot of the Lyapunov exponent λ. Near the parameter resonance at excitation frequency $\Omega = 2\omega_0$. The Lyapunov exponents increase, reaching their maximum values in the center of the instability region. In the deterministic case when $\beta = 0$, it is well known that the Lyapunov exponent of system (2) is $\lambda = -h/2$. So the trivial solution of Eq. (2) is stable if and only if $h > 0$. From Fig. 1(a) and (b), it can be shown that λ is a decreasing function of $|\sigma|$, and reaches its maximum value when $\sigma = 0$. On the other hand λ is a increasing function of β, which means that trivial solution will lose its stability and become unstable as the amplitude of random parameter excitation increase. In short, trivial solution stability of Eq. (2) is as same as system without term x^3. So nontrivial solution steady-state response of Eq. (2) need to study.

3.2. Nontrivial steady-state response

For nontrivial steady-state response $a \neq 0$, Eq. (12) can be written as the following Ito equations:

$$\mathrm{d}a = \left(-\frac{h}{2}a - \frac{\beta}{4\omega_0}a\sin\eta\right)\mathrm{d}T_1,$$

$$\mathrm{d}\eta = \left(\sigma - \frac{\beta}{2\omega_0}\cos\eta - \frac{3\alpha}{\omega_0}a^2\right)\mathrm{d}T_1 + \gamma\,\mathrm{d}W(T_1). \qquad (19)$$

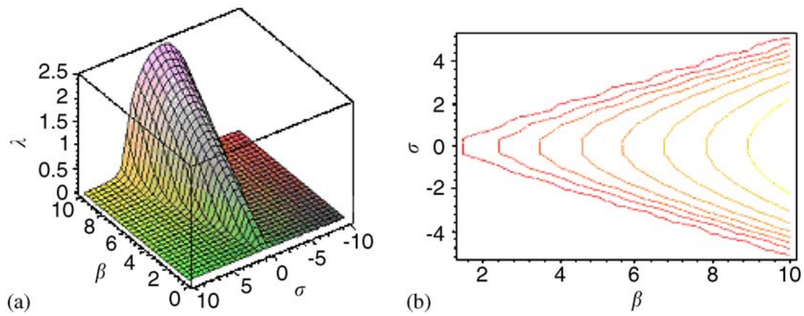

Fig. 1. (a) The Lyapunov exponent and (b) the corresponding level lines.

In the case when γ is small, i.e., $W(T_1)$ is a narrow-band random process, we can use perturbation method to solve Eq. (19). When $\gamma = 0$, the steady-state solution of Eqs. (19) is given by

$$a_0 = \frac{\sqrt{2\sigma\omega_0} - \sqrt{\beta^2 - 4\omega_0^2 h^2}}{\sqrt{6\alpha}},$$

$$\sin \eta_0 = -2h\omega_0/\beta. \tag{20}$$

When $\gamma \neq 0$ but is small, let

$$a = a_0 + a_1, \quad \eta = \eta_0 + \eta_1, \tag{21}$$

where a_0, η_0 are defined by Eqs. (20) and a_1, η_1 are small terms. Substituting the above equations into Eqs. (19) and neglecting the nonlinear terms, we obtain the linearized equations

$$a_1' = -\frac{\beta}{4\omega_0} a_0 \cos \eta_0 \cdot \eta_1,$$

$$\eta_1' = -h\eta_1 - \frac{6\alpha a_0 a_1}{\omega_0} + \gamma W'. \tag{22}$$

The above equations yield

$$a_1'' + h a_1' - \frac{3\alpha\beta a_0 \cos \eta_0}{2\omega_0} a_1 = \frac{\beta\gamma a_0}{4\omega_0} \cos \eta_0 \xi(T_1), \tag{23}$$

where $\xi(T_1) = W'(T_1)$. Since $W(T_1)$ is a standard Wiener process, its form derivative $\xi(T_1)$ is a standard Gaussian white noise. It is clear that (a_1, a_1') are also Gaussian processes, their steady-state probability density function obtained by detail balance method is given as

$$p(a_1', a_1) = C \exp\left\{ -\frac{16h\omega_0^2}{(\beta\gamma a_0 \cos \eta_0)^2} \left[(a_1')^2 - \frac{3\beta\alpha a_0 \cos \eta_0}{2\omega_0} a_1^2 \right] \right\}, \tag{24}$$

where C is the normalization constant. From Eq. (24) it is clear that C can be normalized if and only if $\alpha a_0 \cos \eta_0 < 0$, which is also the necessary and sufficient condition of the existence of nontrivial of Eqs. (11). The steady-state probability density function is given by

$$p(a_1) = C' \exp\left\{ -\frac{24h\omega_0\alpha}{\beta\gamma^2 a_0 \cos \eta_0} a_1^2 \right\}, \tag{25}$$

where C' is the normalization constant. Then the second moment of a_1 is

$$E(a_1^2) = \frac{\sqrt{\pi}}{2} \left(-\frac{\beta\gamma^2 a_0 \cos \eta_0}{24h\omega_0\alpha} \right)^{3/2}. \tag{26}$$

From Eq. (24) one has $E[a_1] = 0$,

$$E(a) = E(a_0 + a_1) = a_0,$$

$$E(a^2) = a_0^2 + \frac{\sqrt{\pi}}{2} \left(-\frac{\beta\gamma^2 a_0 \cos \eta_0}{24h\omega_0\alpha} \right)^{3/2}. \tag{27}$$

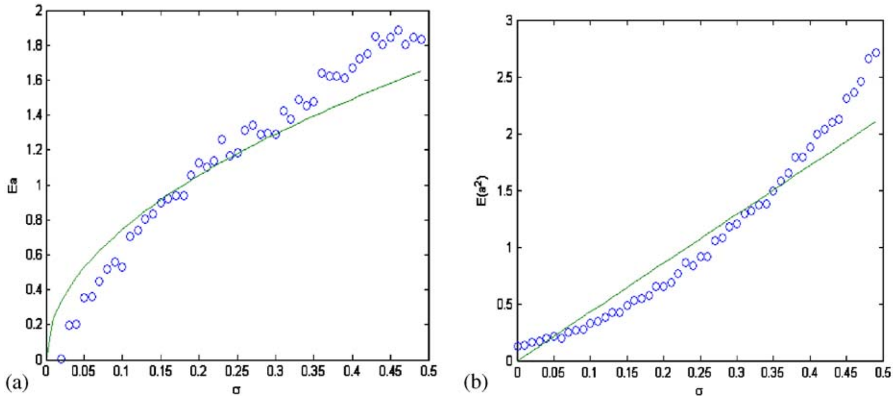

Fig. 2. (a) and (b) — theoretical solution and ○ ○ ○ numerical solution.

3.3. Numerical simulation

For numerical simulation it is more convenient to use the pseudorandom signal. Let $h = 0.1$, $\omega_0 = 1.0$, $\gamma = 0.5$, $\alpha = 0.1$. When $\beta = 0.2$ the numerical results of the first and second moments of Eq. (1) with σ are shown in Fig. 2(a) and (b).

4. Conclusions

The principal resonance of Mathieu–Duffing systems under random parametric excitation is investigated. The behavior, stability be studied by means of qualitative analysis. The effects of damping, detuning, cubic term, and magnitudes of random excitation are analyzed. The contributions from damping and stiffness can be taken fully into account. The theoretical analyses are verified by numerical results. Theoretical analyses and numerical simulations show that when the intensity of the random excitation increases, the trivial steady-state solution loses its stability and then the system may have a nontrivial state solution.

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